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Irreversibility, Uncertainty, and Investment

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Irreversible investment is especially sensitive to such risk factors as volatile exchange rates and uncertainty about tariff structures and future cash flows. If the goal of macroeconomic policy is to stimulate investment, stability and credibility may be more important than tax incentives or interest rates.

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Most major investment expenditures are at least partly irreversible: the firm cannot disinvest, so the expenditures are sunk costs.

Irreversibility has important implications for investment decisions — and for the factors most likely to affect investment spending.

Irreversible investment is especially sensitive to such risk factors as uncertainty about future cash flows, interest rates, and the cost and timing of investments.

Pindyck reviews some simple models of irreversible investment — to explain how optimal investment rules can be obtained from contingent claim analysis or from dynamic programming.

He also discusses how to model investment when irreversibility is important, so one can understand the likely response of investment spending to policy incentives and other changes in the environment.

To the extent that the goal of macroeconomic policy is to stimulate investment, for example, stability and credibility may be more important than tax incentives or interest rates. As a determinant of aggregate investment spending, the level of interest rates may be less important than their volatility (and the volatility of other variables).

Trade reform, when suspected to be only temporary, can also be counterproductive, with aggregate investment declining because of liberalization. Uncertainty about future tariff structures, and hence over future factor returns, creates an opportunity cost for committing capital to new physical plant. Foreign exchange and liquid assets held abroad involve no such commitment, and so may be preferable even though the expected rate of return is lower.

Likewise, it may be difficult to stem or reverse capital flight if the perception is that it may become more difficult to take capital out of the country than to bring it in.

Investments in the energy field may be influenced by the threat of price controls, windfall profit taxes, or related policies that might be imposed should prices rise substantially.

Policies that stabilize prices may influence investment decisions in markets for commodities (such as oil) for which prices are often volatile.

Increases in the volatility of interest or exchange rates depress investment — but it is not clear how much. Determining the importance of these factors — through empirical studies and simulation models — should be a research priority.

This paper is a product of the Macroeconomic Adjustment and Growth Division, Country Economics Department. Copies are available free from the World Bank, 1818 H Street NW, Washington DC 20433. Please contact Nancy Carolan, room N11-037, extension 61737 (53 pages with figures).

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IRREVERSIBILITY, UNCERTAINTY, AND INVESTMENT

1. Introduction.

Despite its importance to economic growth and the evolution of market structure, the investment behavior of firms, industries, and countries remains poorly understood. Econometric models have been notorious in their failure to explain and predict changes in investment spending, and economists have been unable to provide a clear and convincing explanation of why it is that some countries or industries invest more than others. We also lack answers to normative questions. It is difficult, for example, to design tax policies when the dependence of investment on tax rates is not known. And business school students are often misled when they are told to base investment decisions on a simple net present value rule.

Part of the problem may be that most econometric models of investment behavior are based on the implicit assumption that investment expenditures are reversible, i.e., can be "undone." So, too, is the net present value rule as it is usually taught to students in business school: "Invest in a project when the present value of its expected cash flows is at least as large as its cost." This rule is incorrect if the investment is not reversible and the decision to invest can be postponed.

Most major investment expenditures have two important characteristics which together can dramatically affect the decision to invest. First, the expenditures are largely irreversible; the firm cannot disinvest, so the expenditures must be viewed as sunk costs. Second, most major investments can be delayed, giving the firm an opportunity to wait for new information to arrive about prices, costs, and other market conditions before it commits resources.

Irreversibility usually arises because capital is industry or firm specific, i.e., it cannot be used productively in a different industry or by a different firm. A steel plant, for example, is industry specific. It can only be used to produce steel, so if the demand for steel falls, the market value of the plant will fall. Although the plant could be sold to another steel company, there is likely to be little gain from doing so, so the investment in the plant must be viewed as a sunk cost. As another example, most investments in marketing and advertising are firm specific, and so are likewise sunk costs. Partial irreversibility can also result from the "lemons" problem. Office equipment, cars, trucks, and computers are not industry specific, but have resale value well below their purchase cost, even if new.

Irreversibility can also arise because of government regulations or institutional arrangements. For example, capital controls may make it impossible for foreign (or domestic) investors to sell assets and reallocate their funds. Or, investments in new workers may be partly irreversible because of high costs of hiring and firing.¹

As an emerging literature has demonstrated, and as will be explained in this paper, irreversibility undermines the theoretical foundation of standard neoclassical investment models, and also invalidates the NPV rule as it is commonly taught in business schools. Irreversibility also has important implications for the factors that are most likely to affect investment spending. Irreversible investment is especially sensitive to risk factors; for example, uncertainties over the future product prices and

¹I will focus mostly on investment in capital equipment, but the issues that I will discuss also arise in labor markets, as Dornbusch (1987) has pointed out. For a model of how hiring and firing costs affect employment, see Bentolila and Bertola (1988).

operating costs that determine cash flows, uncertainty over future interest rates, and uncertainty over the cost and timing of the investment itself. In the context of macroeconomic policy, this means that if the goal is to stimulate investment, stability and credibility may be much more important than tax incentives or interest rates.

An irreversible investment opportunity is akin to a financial call option. A call option gives the holder the right (for some specified amount of time) to pay an exercise price and in return receive an asset (e.g., a share of stock) that has some value. A firm with an investment opportunity has the option to spend money (the "exercise price") now or in the future, in return for an asset (e.g., a project) of some value. As with a financial call option, the firm's option to invest is valuable in part because the future value of the asset that the firm gets by investing is uncertain. If the asset rises in value, the payoff from investing rises. If it falls in value, the firm need not invest, and will only lose what it spent to obtain the investment opportunity.

How do firms obtain investment opportunities? Sometimes they result from patents, or ownership of land or natural resources. More generally, they arise from a firm's managerial resources, technological knowledge, reputation, market position, and possible scale, all of which may have been built up over time, and which enable the firm to productively undertake investments that individuals or other firms cannot undertake. Most important, these options to invest are valuable. Indeed, for most firms, a substantial part of their market value is attributable to their options to

invest and grow in the future, as opposed to the capital that they already have in place.²

When a firm makes an irreversible investment expenditure, it exercises, or "kills," its option to invest. It gives up the possibility of waiting for new information to arrive that might affect the desirability or timing of the expenditure; it cannot disinvest should market conditions change adversely. This lost option value must be included as part of the cost of the investment. As a result, the NPV rule "Invest when the value of a unit of capital is at least as large as the purchase and installation cost of the unit" is not valid. The value of the unit must exceed the purchase and installation cost, by an amount equal to the value of keeping the option to invest these resources elsewhere alive -- an opportunity cost of investing.

Recent studies have shown that this opportunity cost can be large, and investment rules that ignore it can be grossly in error.³ Also, this opportunity cost is highly sensitive to uncertainty over the future value of the project, so that changing economic conditions that affect the perceived riskiness of future cash flows can have a large impact on investment spending, larger than, say, a change in interest rates. This may explain why neoclassical investment theory has failed to provide good empirical models of investment behavior.

²The importance of growth options as a source of firm value is discussed in Myers (1977). Also, see Kester (1984) and Pindyck (1988).

³See, for example, McDonald and Siegel (1986), Brennan and Schwartz (1985), Majd and Pindyck (1987), and Pindyck (1988). Bernanke (1983) and Cukierman (1980) have developed related models in which firms have an incentive to postpone irreversible investments so that they can wait for new information to arrive. However, in their models, this information makes the future value of an investment less uncertain; we will focus on situations in which information arrives over time, but the future is always uncertain.

This paper has several objectives. First, I will review some basic models of irreversible investment to illustrate the option-like characteristics of investment opportunities, and to show how optimal investment rules can be obtained from methods of option pricing, or alternatively from dynamic programming. Besides demonstrating a methodology that can be used to solve investment problems, this will serve to show how the resulting investment rules depend on various parameters that come from the market environment.

Second, I will discuss the implications of irreversibility for the empirical analysis of investment behavior. At issue is how we can best go about modelling investment when irreversibility is important, so that we can better understand the likely response of investment spending to policy incentives and other changes in the macroeconomic environment.

Finally, I will briefly discuss some of the implications that the irreversibility of investment may have for policy. For example, policies that stabilize prices or exchange rates may be effective ways of stimulating investment. Similarly, a major cost of political and economic instability may be its depressing effect on investment. This is likely to be particularly important for the developing economies. For many LDC's, investment as a fraction of GDP has fallen dramatically during the 1980's, despite moderate economic growth. Yet the success of macroeconomic policy in these countries requires increases in private investment. This has created a Catch-22 that makes the social value of investment higher than its private value. The reason is that if firms do not have confidence that macro policies will succeed and growth trajectories will be maintained, they are afraid to invest, but if they do not invest, macro policies are indeed doomed to fail. It is therefore important to understand how investment

might depend on risk factors that are at least partly under government control, e.g., price, wage, and exchange rate stability, the threat of price controls or expropriation, and changes in trade regimes.

The next section uses a simple two-period example to illustrate how irreversibility can affect an investment decision, and how option pricing methods can be used to value a firm's investment opportunity, and determine whether or not the firm should invest.

Section 3 works through a basic continuous time model of irreversible investment that was first examined by McDonald and Siegel (1986). Here a firm must decide when to invest in a project whose value follows a random walk. We first solve this problem using option pricing methods and then by dynamic programming, and show how the two approaches are related. Section 4 extends this model so that the price of the firm's output follows a random walk, and the firm can (temporarily) stop producing if price falls below variable cost. We will show how both the value of the project and the value of the firm's option to invest in the project can be determined, and derive the optimal investment rule and examine its properties.

Section 5 discusses some of the empirical issues that arise when investment is irreversible. We will argue that traditional approaches to modelling aggregate investment spending are unlikely to be successful, briefly discuss some tests that might be carried out to determine the importance of irreversibility. Finally, Section 6 discusses policy implications, and suggests future research.

2. A Simple Two-Period Example.

The implications of irreversibility and the option-like nature of an investment opportunity can be demonstrated most easily with a simple two-

period example. Consider a firm's decision to irreversibly invest in a widget factory. The factory can be built instantly, at a cost I , and will produce one widget per year forever, with zero operating cost. Currently the price of widgets is \$100, but next year the price will change. With probability q , it will rise to \$150, and with probability $(1-q)$ it will fall to \$50. The price will then remain at this new level forever. (See Figure 1.) We will assume that this risk is fully diversifiable, so that the firm can discount future cash flows using the risk-free rate, which we will take to be 10 percent.

For the time being we will set $I = \$800$ and $q = .5$. Is this a good investment? Should we invest now, or wait one year and see whether the price goes up or down? Suppose we invest now. Calculating the net present value of this investment in the standard way, we get:

$$NPV = -800 + \sum_{t=1}^{\infty} 100/(1.1)^t = -800 + 1,100 = \underline{\$300}$$

The NPV is positive; the current value of a widget factory is $V_0 = 1,100 > 800$. Hence it would seem that we should go ahead with the investment.

This conclusion is incorrect, however, because the calculations above ignore a cost - the opportunity cost of investing now, rather than waiting and thereby keeping open the possibility of not investing should the price fall. To see this, calculate the NPV of this investment opportunity, assuming we wait one year and then invest if the price goes up:

$$NPV = (.5)[-800/1.1 + \sum_{t=1}^{\infty} 150/(1.1)^t] = 425/1.1 = \underline{\$386}$$

(Note that in year 0, there is no expenditure and no revenue. In year 1, the 800 is spent only if the price rises to \$150, which will happen with

probability .5.) The NPV today is higher if we plan to wait a year, so clearly waiting is better than investing now.

Note that if our only choices were to invest today or never invest, we would invest today. In that case there is no option to wait a year, and hence no opportunity cost to killing such an option, so the standard NPV rule would apply. Two things are needed to introduce an opportunity cost into the NPV calculation - irreversibility, and the ability to invest in the future as an alternative to investing today. There are, of course, situations in which a firm cannot wait, or cannot wait very long, to invest. (One example is the anticipated entry of a competitor into a market that is only large enough for one firm. Another example is a patent or mineral resource lease that is about to expire.) The less time there is to delay, and the greater the cost of delaying, the less will irreversibility affect the investment decision. We will develop this point again in Section 3 in the context of a more general model.

How much is it worth to have the flexibility to make the investment decision next year, rather than having to invest either now or never? (We know that having this flexibility is of some value, because we would prefer to wait rather than invest now.) The value of this "flexibility option" is easy to calculate; it is just the difference between the two NPV's, i.e., $\$386 - \$300 = \$86$.

Finally, suppose there exists a futures market for widgets, with the futures price for delivery one year from now equal to the expected future spot price, i.e., $\$100$.⁴ Would the ability to hedge on the futures market

⁴In this example, the futures price would equal the expected future price because we assumed that the risk is fully diversifiable. (If the price of widgets were positively correlated with the market portfolio, the
(continued...)

change our investment decision? Specifically, would it lead us to invest now, rather than waiting a year? The answer is no. To see this, note that if we were to invest now, we would hedge by selling short futures for 5 widgets; this would exactly offset any fluctuations in the NPV of our project next year. But this would also mean that the NPV of our project today is \$300, exactly what it is without hedging. Hence there is no gain from hedging (the risk is diversifiable), and we are still better off waiting until next year to make our investment decision.

Analogy to Financial Options.

Our investment opportunity is analogous to a call option on a common stock. It gives us the right (which we need not exercise) to make an investment expenditure (the exercise price of the option) and receive a project (a share of stock) the value of which fluctuates stochastically. In the case of our simple example, next year if the price rises to \$150, we exercise our option by paying \$800 and receive an asset which will be worth $V_1 = \$1650 (= \Sigma_0^{\infty} 150/1.1^t)$. If the price falls to \$50, this asset will be worth only \$550, and so we will not exercise the option. We found that the value of our investment opportunity (assuming that the actual decision to invest can indeed be made next year) is \$386. It will be helpful to recalculate this value using option pricing methods, because later we will use such methods to analyze other investment problems.

⁴(...continued)

futures price would be less than the expected future spot price.) Note that if widgets were storable and aggregate storage is positive, the marginal convenience yield from holding inventory would then be 10 percent. The reason is that since the futures price equals the current spot price, the net holding cost (the interest cost of 10 percent less the marginal convenience yield) must be zero.

To do this, let F_0 denote the value today of the investment opportunity, i.e., what we should be willing to pay today to have the option to invest in the widget factory, and let F_1 denote its value next year. Note that F_1 is a random variable; it depends on what happens to the price of widgets. If the price rises to \$150, then $F_1 = \sum_0^\infty 150/(1.1)^t = 800 = \850 . If the price falls to \$50, the option to invest will go unexercised, so that $F_1 = 0$. Thus we know all possible values for F_1 . The problem is to find F_0 , the value of the option today.

To solve this problem, we will create a portfolio that has two components: the investment opportunity itself, and a certain number of widgets. We will pick this number of widgets so that the portfolio is risk-free, i.e., so that its value next year is independent of whether the price of widgets goes up or down. Since the portfolio will be risk-free, we know that the rate of return one can earn from holding it must be the risk-free rate. By setting the portfolio's return equal to that rate, we will be able to calculate the current value of the investment opportunity.

Specifically, consider a portfolio in which one holds the investment opportunity, and sells short n widgets.⁵ The value of this portfolio today is $\Phi_0 = F_0 - nP_0 = F_0 - 100n$. The value next year, $\Phi_1 = F_1 - nP_1$, depends on P_1 . If $P_1 = 150$ so that $F_1 = 850$, $\Phi_1 = 850 - 150n$. If $P_1 = 50$ so that $F_1 = 0$, $\Phi_1 = -50n$. Now, let us choose n so that the portfolio is risk-free, i.e., so that Φ_1 is independent of what happens to price. To do this, just set:

$$850 - 150n = -50n,$$

⁵If widgets were a traded commodity (like oil), one could obtain a short position by borrowing from another producer, or by going short in the futures market. For the moment, however, we need not be concerned with the actual implementation of this portfolio.

or, $n = 8.5$. With n chosen this way, $\Phi_1 = -425$, whether the price goes up or down.

We now calculate the return from holding this portfolio. That return is the capital gain, $\Phi_1 - \Phi_0$, minus any payments that must be made to hold the short position. Since the expected rate of capital gain on a widget is zero (the expected price next year is \$100, the same as this year's price), no rational investor would hold a long position unless he or she could expect to earn at least 10 percent. Hence selling widgets short will require a payment of $.1P_0 = \$10$ per widget per year.⁶ Our portfolio has a short position of 8.5 widgets, so it will have to pay out a total of \$85. The return from holding this portfolio over the year is thus $\Phi_1 - \Phi_0 - 85 = -425 - F_0 + 850 - 85 = 340 - F_0$.

Because this return is risk-free, we know that it must equal the risk-free rate, which we have assumed is 10 percent, times the initial value of the portfolio, $\Phi_0 = F_0 - nP_0$:

$$340 - F_0 = .1(F_0 - 850),$$

We can thus determine that $F_0 = \$386$. Note that this is the same value that we obtained before by calculating the NPV of the investment opportunity under the assumption that we follow the optimal strategy of waiting a year before deciding whether to invest.

We have found that the value of our investment opportunity, i.e., the value of the option to invest in this project, is \$386. The payoff from investing (exercising the option) today is $\$1100 - \$800 = \$300$. But once we invest, our option is gone, so the \$386 is an opportunity cost of investing.

⁶This is analogous to selling short a dividend-paying stock. The short position requires payment of the dividend, because no rational investor will hold the offsetting long position without receiving that dividend.

Hence the full cost of the investment is $\$800 + \$386 = \$1186 > \1100 . As a result, we should wait and keep our option alive, rather than invest today. We have thus come to the same conclusion as we did by comparing NPV's. This time, however, we calculated the value of the option to invest, and explicitly took it into account as one of the costs of investing.

Our calculation of the value of the option to invest was based on the construction of a risk-free portfolio, which requires that one can trade (hold a long or short position in) widgets. Of course, we could just as well have constructed our portfolio using some other asset, or combination of assets, the price of which is perfectly correlated with the price of widgets. But what if one cannot trade widgets, and there are no other assets that "span" the risk in a widget's price? In this case one could still calculate the value of the option to invest the way we did at the outset - by computing the NPV for each investment strategy (invest today versus wait a year and invest if the price goes up), and picking the strategy that yields the highest NPV. That is essentially the dynamic programming approach. In this case it gives exactly the same answer, because all price risk is diversifiable. In Section 3 we will explore this connection between option pricing and dynamic programming in more detail.

Changing the Parameters.

So far we have fixed the direct cost of the investment, I , at \$800. We can obtain further insight by changing this number, as well as other parameters, and calculating the effects on the value of the investment opportunity and on the investment decision. For example, by going through the same steps as above, it is easy to see that the short position needed to obtain a risk-free portfolio depends on I as follows:

$$n = 16.5 - .01I$$

The current value of the option to invest is then given by:

$$F_0 = 750 - .455I$$

The reader can check that as long as $I > \$642$, F_0 exceeds the net benefit from investing today (rather than waiting), $V_0 - I = \$1,100 - I$. Hence if $I > \$642$, one should wait rather than invest today. However, if $I = \$642$, $F_0 = \$458 = V_0 - I$, so that one would be indifferent between investing today and waiting until next year. (This can also be seen by comparing the NPV of investing today with the NPV of waiting until next year.) And if $I < \$642$, one should invest today rather than wait. The reason is that in this case the lost revenue from waiting exceeds the opportunity cost of closing off the option of waiting and not investing should the price fall. This is illustrated in Figure 2, which shows the value of the option, F_0 , and the net payoff, $V_0 - I$, both as functions of I . For $I > \$642$, $F_0 = 750 - .455I > V_0 - I$, so the option should be kept alive. However, if $I < \$642$, $750 - .455I < V_0 - I$, so the option should be exercised, and hence its value is just the net payoff, $V_0 - I$.

We can also determine how the value of the investment option depends on q , the probability that the price of widgets will rise next year. To do this, let us once again set $I = \$800$. The reader can verify that the short position needed to obtain a risk-free portfolio is independent of q , i.e., is $n = 8.5$. The payment required for the short position, however, does depend on q , because the expected capital gain on a widget depends on q . The expected rate of capital gain is $[E(P_1) - P_0]/P_0 = q - .5$, so the required payment per widget in the short position is $.1 - (q - .5) = .6 - q$. By following the same steps as above, the reader can check that the value today of the option to invest is $F_0 = 773q$. This can also be written as a function of the current value of the project, V_0 . We have $V_0 = 100 +$

$\sum_1^{\infty} (100q + 50)/(1.1)^t = 600 + 1000q$, so $F_0 = .773V_0 - 464$. Finally, note that it is better to wait rather than invest today as long as $F_0 > V_0 - I$, or $q < .88$.

There is nothing special about the particular source of uncertainty that we introduced in this problem. There will be a value to waiting (i.e., an opportunity cost to investing today rather than waiting for information to arrive) whenever the investment is irreversible and the net payoff from the investment evolves stochastically over time. Thus we could have constructed our example so that the uncertainty arises over future exchange rates, factor input costs, or government policy. For example, the payoff from investing, V , might rise or fall in the future depending on (unpredictable) changes in policy. Alternatively, the cost of the investment, I , might rise or fall, in response to changes in materials costs, or to a policy change, such as the granting or taking away of an investment subsidy or tax benefit.

In our example, we made the unrealistic assumption that there is no longer any uncertainty after the second period. Instead, we could have allowed the price to change unpredictably each period. For example, we could posit that at $t = 2$, if the price is \$150, it could increase to \$225 with probability q or fall to \$75 with probability $(1-q)$, and if it is \$50 it could rise to \$75 or fall to \$25. Price could rise or fall in a similar way at $t = 3, 4$, etc. One could then work out the value of the option to invest, and the optimal rule for exercising that option. Although the algebra is messier, the method is essentially the same as for the simple two-period exercise we carried out above.⁷ Rather than take this approach,

⁷This is the basis for the binomial option pricing model. See Cox and Rubinstein (1985) for a detailed discussion.

in the next section we extend our example by allowing the payoff from the investment to fluctuate continuously over time.

3. A More General Problem of Investment Timing.

One of the more basic models of irreversible investment is that of McDonald and Siegel (1986). They considered the following problem: At what point is it optimal to pay a sunk cost I in return for a project whose value is V , given that V evolves according to:

$$dV/V = \alpha dt + \sigma dz \quad (1)$$

where dz is the increment of a Wiener process.⁸ Eqn. (1) implies that the current value of the project is known, but future values are lognormally distributed with a variance that grows linearly with the time horizon. And although information arrives over time (the firm observes V changing), the future value of the project is always uncertain.⁹

⁸That is, $dz = \epsilon(t)(dt)^{1/2}$, with $\epsilon(t)$ a serially uncorrelated and normally distributed random variable. I will assume that the reader is familiar with continuous-time stochastic processes, the use of Ito's Lemma, and stochastic dynamic programming. For an introduction to these methods, see Merton (1971), the Appendix to Fischer (1975), or Malliaris and Brock (1982).

⁹This process for V can be viewed as a special case of the more general mean-reverting process:

$$dV/V = [\alpha + \lambda(V^* - V)]dt + \sigma dz \quad (i)$$

where V^* is a mean or "normal" value to which V tends to revert, and λ measures the speed of this reversion. (Eqn. (i) is the continuous-time version of a first-order autoregressive process.) If $\lambda = 0$, then (i) becomes eqn. (1), i.e., V is a random walk. At the opposite extreme, as λ is made very large, V approaches V^* plus a serially uncorrelated error, i.e., shocks to V this period have no effect on next period's V . We will focus on the random walk version of eqn. (1) because it is analytically convenient, and because it makes it easier to understand the effects of uncertainty.

McDonald and Siegel pointed out that the investment opportunity is equivalent to a perpetual call option, and deciding when to invest is equivalent to deciding when to exercise such an option. Thus, the investment decision can be viewed as a problem of option valuation (as we saw in the simple example presented in the previous section). I will re-derive the solution to their problem in two ways, first using option pricing (contingent claims) methods, and then via dynamic programming. This will allow us to compare these two approaches and the assumptions that each requires. We will then examine the characteristics of the solution.

Option Pricing Approach.

As we have seen, the firm's option to invest, i.e., its option to pay I and receive a project worth V , is analogous to a call option on a stock. Unlike most financial call options, it is a perpetual option -- it has no expiration date. We can value this option and determine the optimal exercise (investment) rule using the same methods that are used to value financial options.¹⁰ To do this we need to make one important assumption.

We must assume that changes in V are spanned by existing assets. Specifically, it must be possible to find an asset or construct a dynamic portfolio of assets the price of which is perfectly correlated with V .¹¹ This is equivalent to saying that markets are sufficiently complete that the firm's decisions do not affect the opportunity set available to investors. The assumption of spanning should hold for most commodities, which are typically traded on both spot and futures markets, and for manufactured

¹⁰For an overview of contingent claims methods and their application, see Cox and Rubinstein (1985) and Mason and Merton (1985).

¹¹Note that a dynamic portfolio is a portfolio whose holdings are adjusted continuously as asset prices change.

goods to the extent that prices are correlated with the values of shares or portfolios. However, in some cases this assumption will not hold, for example, a new product unrelated to any existing ones.

With the spanning assumption, we can determine the investment rule that maximizes the firm's market value without making any assumptions about risk preferences or discount rates, and the investment problem reduces to one of contingent claim valuation. (We will see shortly that if spanning does not hold, dynamic programming can still be used to maximize the present value of the firm's expected flow of profits, subject to an arbitrary discount rate.)

Let x be the price of an asset or portfolio of assets perfectly correlated with V , and denote by ρ_{V_m} the correlation of V with the market portfolio. Then x evolves according to:

$$dx = \mu x dt + \sigma x dz,$$

and by the CAPM, its expected return is $\mu = r + \phi \rho_{V_m} \sigma$, where ϕ is the market price of risk. We can assume that α , the expected percentage rate of change of V , is less than its risk-adjusted return μ , because as will become clear, if this were not the case, the firm would never invest. No matter what the current level of V , the firm would always be better off waiting and simply holding on to the option to invest. We denote by δ the difference between μ and α , i.e., $\delta = \mu - \alpha$.

A few words about the meaning of δ are in order, given the important role it plays in this model. The analogy with a financial call option is helpful here. If V were the price of a share of common stock, δ would be the dividend rate on the stock. The total expected return on the stock would be $\mu - \delta + \alpha$, i.e., the dividend rate plus the expected rate of capital gain.

If the dividend rate δ were zero, a call option on the stock would always be held to maturity, and never exercised prematurely. The reason is that the entire return on the stock is captured in its price movements, and hence by the call option, so there is no cost to keeping the option alive. But if the dividend rate is positive, there is an opportunity cost to keeping the option alive rather than exercising it. That opportunity cost is the dividend stream that one foregoes by holding the option rather than the stock. Since δ is a proportional dividend rate, the higher the price of the stock, the greater the flow of dividends. At some high enough price, the opportunity cost of foregone dividends becomes high enough to make it worthwhile to exercise the option.

For our investment problem, μ is the expected rate of return from owning the completed project. It is the equilibrium rate established by the capital market, and includes an appropriate risk premium. If $\delta > 0$, the expected rate of capital gain on the project is less than μ . Hence δ is an opportunity cost of delaying construction of the project, and instead keeping the option to invest alive. If δ were zero, there would be no opportunity cost to keeping the option alive, and one would never invest, no matter how high the NPV of the project. That is why we assume $\delta > 0$. On the other hand, if δ is very large, the value of the option will be very small, because the opportunity cost of waiting is large. As $\delta \rightarrow \infty$, the value of the option goes to zero; in effect, the only choices are to invest now or never, and the standard NPV rule will again apply.

The parameter δ can be interpreted in different ways. For example, it could reflect the process of entry and capacity expansion by competitors. Or it can simply reflect the cash flows from the project. If the project is infinitely lived, then eqn. (1) can represent the evolution of V during the

operation of the project, and δV is the rate of cash flow that the project yields. Since we assume δ is constant, this is consistent with future cash flows being a constant proportion of the project's market value.¹²

Eqn. (1) is, of course, is an abstraction from most real projects. For example, if variable cost is positive and the project can be shut down temporarily when price falls below variable cost, V will not follow a lognormal process, even if the output price does. Nonetheless, eqn. (1) is a useful simplification that will help to clarify the main effects of irreversibility and uncertainty. We will discuss more complicated (and hopefully more realistic) models later.

Solving the Investment Problem.

Let us now turn to the valuation of our investment opportunity, and the optimal investment rule. Let $F = F(V)$ be the value of the firm's option to invest. To find $F(V)$ and the optimal investment rule, we consider the following hedge portfolio: long the option worth $F(V)$, and short dF/dV units of the project (or equivalently, of the asset or portfolio x). Using subscripts to denote derivatives, the value of this portfolio is $P = F - F_V V$.

The short position in this portfolio will require a payment of $\delta V F_V$ dollars per time period; otherwise no rational investor will enter into the

¹²A constant payout rate, δ , and required return, μ , imply an infinite project life. Letting CF denote the cash flow from the project:

$$V_0 = \int_0^T CF_t e^{-\mu t} dt = \int_0^T \delta V_0 e^{(\mu - \delta)t} e^{-\mu t} dt ,$$

which implies $T = \infty$. If the project has a finite life, then eq. (1) cannot represent the evolution of V during the operating period. However, it can represent its evolution prior to construction of the project, which is all that matters for the investment decisions. See Majd and Pindyck (1987), pp. 11 - 13, for a detailed discussion of this point.

long side of the transaction. Taking this into account, the total instantaneous return from holding the portfolio is:

$$dF - F_V dV - \delta V F_V dt$$

We will see that this return is risk-free, and therefore must equal $r(F - F_V V)dt$:

$$dF - F_V dV - \delta V F_V dt = r(F - F_V V)dt \quad (2)$$

To obtain an expression for dF , use Ito's Lemma:

$$dF = F_V dV + (1/2)F_{VV}(dV)^2 \quad (3)$$

(Higher order terms go to zero.) Substitute (1) for dV , with $\alpha = \mu - \delta$, and $(dV)^2 = \sigma^2 V^2 dt$ into eqn. (3):

$$dF = (\mu - \delta)V F_V dt + \sigma V F_V dz + (1/2)\sigma^2 V^2 F_{VV} dt \quad (4)$$

Now substitute (4) into (2), rearrange terms, and note that all terms in dz cancel out, so the portfolio is indeed risk-free:

$$(1/2)\sigma^2 V^2 F_{VV} + (r - \delta)V F_V - rF = 0 \quad (5)$$

Eqn. (5) is a differential equation that $F(V)$ must satisfy. In addition, $F(V)$ must satisfy the following boundary conditions:

$$F(0) = 0 \quad (6a)$$

$$F(V^*) = V^* - I \quad (6b)$$

$$F_V(V^*) = 1 \quad (6c)$$

Condition (6a) says that if V goes to zero, it will stay at zero (an implication of the process (1)), so the option to invest will be of no value. V^* is the price at which it is optimal to invest, and (6b) just says that upon investing, the firm receives a net payoff $V^* - I$. Condition (6c) is called the "smooth pasting" or "high contact" condition. If $F(V)$ were not continuous and smooth at the critical exercise point V^* , one could do better by exercising at a different point.¹³

¹³Dixit (1988) provides a nice heuristic explanation of this condition.

To find $F(V)$, we must solve eqn. (5) subject to the boundary conditions (6a-c). In this case we can guess a functional form, and determine by substitution if it works. It is easy to see that the solution to eqn. (5) which also satisfies condition (6a) is:

$$F(V) = aV^\beta \quad (7)$$

where a is a constant, and β is given by:¹⁴

$$\beta = 1/2 - (r-\delta)/\sigma^2 + \{[(r-\delta)/\sigma^2 - 1/2]^2 + 2r/\sigma^2\}^{1/2} \quad (8)$$

The remaining boundary conditions, (6b) and (6c), can be used to solve for the two remaining unknowns: the constant a , and the critical value V^* at which it is optimal to invest. By substituting (7) into (6b) and (6c), it is easy to see that:

$$V^* = \beta I / (\beta - 1) \quad (9)$$

$$\text{and} \quad a = (V^* - I) / (V^*)^\beta \quad (10)$$

Eqns. (7) - (10) give the value of the investment opportunity, and the optimal investment rule, i.e., the critical value V^* at which it is optimal (in the sense of maximizing the firm's market value) to invest. We will examine the characteristic of this solution below. Here we simply point out that we obtained this solution by showing that a hedged (risk-free) portfolio could be constructed consisting of the option to invest and a short position in the project. However, $F(V)$ must be the solution to eqn. (5) even if the option to invest (or the project) does not exist and could

¹⁴The general solution to eqn. (5) is

$$F(V) = a_1 V^{\beta_1} + a_2 V^{\beta_2},$$

where $\beta_1 = 1/2 - (r-\delta)/\sigma^2 + \{[(r-\delta)/\sigma^2 - 1/2]^2 + 2r/\sigma^2\}^{1/2} > 1$,

and $\beta_2 = 1/2 - (r-\delta)/\sigma^2 - \{[(r-\delta)/\sigma^2 - 1/2]^2 + 2r/\sigma^2\}^{1/2} < 0$.

Boundary condition (6a) implies that $a_2 = 0$, so the solution can be written as in eqn. (7).

not be included in the hedge portfolio. All that is required is spanning, i.e., that one could find or construct an asset or portfolio of assets (x) that replicates the stochastic dynamics of V. As Merton (1977) has shown, one can replicate the value function with a portfolio consisting only of the asset x and risk-free bonds, and since the value of this portfolio will have the same dynamics as F(V), the solution to (5), F(V) must be the value function to avoid dominance.

As discussed earlier, spanning will not always hold. If that is the case, one can still solve the investment problem using dynamic programming, as is shown below.

Dynamic Programming.

To solve the problem by dynamic programming, note that we want a rule that maximizes the value of our investment opportunity, F(V):

$$F(V) = \max E_t[(V_T - I)e^{-\mu T}] \quad (11)$$

where E_t denotes the expectation at time t, T is the (unknown) future time that the investment is made, μ is the discount rate, and the maximization is subject to eqn. (1) for V. We will assume that $\mu > \alpha$, and denote $\delta = \mu - \alpha$.

Since the investment opportunity, F(V), yields no cash flows up to the time T that the investment is undertaken, the only return from holding it is its capital appreciation. Hence the Bellman equation for this problem is simply:

$$\mu F = (1/dt)E_t dF \quad (12)$$

Eqn. (12) just says that the total instantaneous return on the investment opportunity, μF , is equal to its expected rate of capital appreciation.

Now expand dF using Ito's Lemma, as in eqn. (3), and substitute (1) for dV and $(dV)^2$:

$$dF = \alpha V F_V dt + \sigma V F_V dz + (1/2)\sigma^2 V^2 F_{VV} dt$$

Since $E_t(dz) = 0$, $(1/dt)E_t dF = \alpha VF_V + (1/2)\sigma^2 V^2 F_{VV}$, and eqn. (12) can be rewritten as:

$$(1/2)\sigma^2 V^2 F_{VV} + \alpha VF_V - \mu F = 0$$

or, substituting $\alpha = \mu - \delta$,

$$(1/2)\sigma^2 V^2 F_{VV} + (\mu - \delta)VF_V - \mu F = 0 \quad (13)$$

Observe that this equation is almost identical to eqn. (5); the only difference is that the discount rate μ replaces the risk-free rate r . The boundary conditions (6a) - (6c) also apply here, and for the same reasons as before. (Note that (6c) follows from the fact that V^* is chosen to maximize the net payoff $V^* - I$.) Hence the contingent claims solution to our investment problem is equivalent to a dynamic programming solution, under the assumption of risk neutrality.¹⁵

Thus if spanning does not hold, we can still obtain a solution to the investment problem, subject to some discount rate. The solution will clearly be of the same form, and the effects of changes in σ or δ will likewise be the same. One point is worth noting, however. Without spanning, there is no theory for determining the "correct" value for the discount rate μ . The CAPM, for example, would not hold either, and so it could not be used to calculate a risk-adjusted discount rate.

¹⁵This result was first demonstrated by Cox and Ross (1976). Also, note that eqn. (5) is the Bellman equation for the maximization of the net payoff to the hedge portfolio that we constructed. Since the portfolio is risk-free, the Bellman equation for that problem is:

$$rP = -\delta VF_V + (1/dt)E_t dP \quad (1)$$

i.e., the return on the portfolio, rP , equals the per period cash flow that it pays out (which is negative, since δVF_V must be paid in to maintain the short position), plus the expected rate of capital gain. By substituting $P = F - F_V V$ and expanding dF as before, one can see that (5) follows from (1).

Characteristics of the Solution.

A few numerical solutions will help to illustrate the results and show how they depend on the values of the various parameters. As we will see, these results are largely the same as those that come out of standard option pricing models. Unless otherwise noted, in what follows we set $r = .04$, $\delta = .04$, and the cost of the investment, I , equal to 1.

Figure 3 shows the value of the firm's investment opportunity, $F(V)$, for $\sigma = 0.2$ and 0.3 . (These values are probably conservative for most projects; in volatile markets, the standard deviation of annual changes in a project's value can easily exceed 20 or 30 percent.) The tangency point of $F(V)$ with the line $V - I$ gives the critical value of V , V^* ; the firm should invest only if $V \geq V^*$. For any positive σ , $V^* > I$. Thus the standard NPV rule, "Invest when the value of a project is at least as great as its cost," is incorrect; it ignores the opportunity cost of investing now rather than waiting for new information. That opportunity cost is exactly $F(V)$. When $V < V^*$, $V < I + F(V)$, i.e., the value of the project is less than its full cost, the direct cost I plus the opportunity cost of "killing" the investment option.

Note that $F(V)$ increases when σ increases, and so too does the critical value V^* . Thus uncertainty increases the value of a firm's investment opportunities, but decreases the amount of actual investing that the firm will do. As a result, when a firm's market or economic environment becomes more uncertain, the market value of the firm can go up, even though the firm does less investing and perhaps produces less! This also makes it easier to understand why as oil prices fell during the mid-1980's but at the same time the perceived uncertainty over future oil prices rose, oil companies

paid more than ever for offshore leases and other oil-bearing lands, even though their development expenditures fell and they produced less.

Finally, note that this effect of uncertainty involves no assumptions about risk preferences, or about the extent to which the riskiness of V is correlated with the market. Firms can be risk-neutral, and stochastic changes in V can be completely diversifiable; an increase in σ will still increase V^* and hence tend to depress investment.

Figures 4 and 5 show how $F(V)$ and V^* depend on δ . Note that an increase in δ from .04 to .08 results in a decrease in $F(V)$, and hence a decrease in the critical value V^* . (In the limit as $\delta \rightarrow \infty$, $F(V) \rightarrow 0$ for $V < I$, and $V^* \rightarrow I$, as Figure 5 shows.) The reason is that as δ becomes larger, the expected rate of growth of V falls, and hence the expected appreciation in the value of the option to invest and acquire V falls. In effect, it becomes costlier to wait rather than invest now. To see this, consider the example of an investment in an apartment building, where δV represents the net flow of rental income. The total return on the building, which must equal the risk-adjusted market rate, has two components - this income flow plus the expected rate of capital gain. Hence the greater is the income flow relative to the total return on the building, the more one forgoes by holding an option to invest in the building, rather than owning the building itself.

If the risk-free rate, r , is increased, $F(V)$ increases, and so does V^* . The reason is that the present value of an investment expenditure I made at a future time T is Ie^{-rT} , but the present value of the project that one receives in return for that expenditure is $Ve^{-\delta T}$. Hence with δ fixed, an increase in r reduces the present value of the cost of the investment but does not reduce its payoff. But note that while an increase in r raises the

value of a firm's investment options, it also results in fewer of those options being exercised. Hence higher (real) interest rates reduce investment, but for a different reason than in the standard model.

4. A Simple Extension.

As explained earlier, eqn. (1) is an abstraction from most real projects. A more realistic model would treat the price of the project's output as a geometric random walk (and possibly one or more factor input costs as well), rather than the value of the project. In addition, it would also account for the fact that the project can be shut down (permanently or temporarily) if price falls below variable cost.

The model developed in the previous section can easily be extended in this way. Specifically, we will assume that the output price, P , follows the stochastic process:

$$dP/P = \alpha dt + \sigma dz \quad (14)$$

where $\alpha < \mu$, the market risk-adjusted rate on P or an asset perfectly correlated with P , and $\delta = \mu - \alpha$ as before. If the output of the project is a storable commodity (e.g., oil or copper), δ will represent the marginal net rate of convenience yield from storage, i.e., the flow of benefits (less storage costs) that the marginal stored unit provides. Note that for simplicity we are assuming that δ is constant; for most commodities, the marginal convenience yield fluctuates over time as the total amount of storage fluctuates.

We will assume for simplicity that marginal and average production cost is equal to a constant, c , and that the project can be costlessly shut down if P falls below c , and later restarted if P rises above c . The project

produces one unit of output per period, and the cost of investing in the project is I . We will also assume that the project is infinitely lived.

We now have two problems to solve. First, we must find the value of this project, $V(P)$. Note that the project represents a set of options (which we will call operating options). Specifically, once the project has been built, the firm has, for each future time t , an option to produce a unit of output, i.e., an option to pay c and receive P . Hence the project is equivalent to a large number (in this case, infinite, because the project is assumed to last indefinitely) of operating options, and can be valued accordingly.¹⁶

Second, given the value of the project, we must value the firm's option to invest in this project, and determine the optimal exercise (investment) rule. This will boil down to finding a critical P^* , where the firm invests only if $P \geq P^*$. Below, we show how the two steps to this problem can be solved sequentially.¹⁷

Valuing the Project.

If we assume that uncertainty over P is spanned by existing assets, we can value the project (and the option to invest in the project) using

¹⁶This point is discussed in detail in McDonald and Siegel (1985). Brennan and Schwartz (1985) developed a model similar to this for a mining project, but is more general in that it allows for costs of shutting down and starting back up. In a related paper, Dixit (1989) developed a model of costly entry and exit in which a firm can decide to become inactive, but if it later wants to produce, it must again incur an entry cost. Hence the firm will produce when price is below variable cost. In the model presented below, the firm can costlessly stop and restart production, and only pays a one-time investment cost when it enters the market.

¹⁷Note that the option to invest is an option to purchase a package of call options (because the project is just a set of options to pay c and receive P at each future time t). Hence we are valuing a compound option. For examples of compound financial option valuation, see Geske (1979) and Carr (1988). Our problem can be treated in a simpler manner.

contingent claim methods. Otherwise, we can specify a discount rate and use dynamic programming. We will assume spanning and use the first approach.

As before, we construct a risk-free portfolio: long the project and short V_P units of the output. This portfolio has value $V(P) - V_P P$, and yields the instantaneous cash flow $j(P-c)dt - \delta V_P P dt$, where $j = 1$ if $P \geq c$ so that the firm is producing, and $j = 0$ otherwise. (Recall that $\delta V_P P dt$ is the payment that must be made to maintain the short position.) The total return on the portfolio is thus $dV - V_P dP + j(P-c)dt - \delta V_P P dt$. Since this return is risk-free, set it equal to $r(V - V_P P)dt$. Expanding dV using Ito's Lemma, substituting (14) for dP , and rearranging yields the following differential equation for V :

$$(1/2)\sigma^2 P^2 V_{PP} + (r-\delta)PV_P - rV + j(P-c) = 0 \quad (15)$$

This equation must be solved subject to the following boundary conditions:

$$V(0) = 0 \quad (16a)$$

$$V(c^-) = V(c^+) \quad (16b)$$

$$V_P(c^-) = V_P(c^+) \quad (16c)$$

$$\lim_{P \rightarrow \infty} V = P/\delta - c/r \quad (16d)$$

Condition (16a) is an implication of eqn. (14), i.e., if P is ever zero it will stay at zero, so the project will then have no value. Condition (16d) says that as P becomes very large, the probability that over any finite time period it will fall below cost and production will cease becomes very small. Hence the value of the project approaches the difference between two perpetuities: a flow of revenue (P) that is discounted at the risk-adjusted rate μ but is expected to grow rate α , and a flow of cost (c), which is constant and certain and hence is discounted at rate r . Finally, conditions

(16b) and (16c) say that the value of the project is a continuous and smooth function of P .

The solution to eqn. (15) will have two parts, one for $P < c$, and one for $P \geq c$. The reader can check by substitution that the following satisfies (15) as well as boundary conditions (16a) and (16d):

$$V(P) = \begin{cases} A_1 P^{\beta_1} & ; P < c \\ A_2 P^{\beta_2} + P/\delta - c/r & ; P \geq c \end{cases} \quad (17)$$

where:¹⁸ $\beta_1 = 1/2 - (r-\delta)/\sigma^2 + \{[(r-\delta)/\sigma^2 - 1/2]^2 + 2r/\sigma^2\}^{1/2}$

and $\beta_2 = 1/2 - (r-\delta)/\sigma^2 - \{[(r-\delta)/\sigma^2 - 1/2]^2 + 2r/\sigma^2\}^{1/2}$.

The constants A_1 and A_2 can be found by applying boundary conditions (16b) and (16c):

$$A_1 = \frac{r - \beta_2(r-\delta)}{r\delta(\beta_1 - \beta_2)} c^{(1-\beta_1)}$$

$$A_2 = \frac{r - \beta_1(r-\delta)}{r\delta(\beta_1 - \beta_2)} c^{(1-\beta_2)}$$

The solution (17) for $V(P)$ can be interpreted as follows. When $P < c$, the project is not producing. Then, $A_1 P^{\beta_1}$ is the value of the firm's option to produce in the future, if and when P increases. When $P \geq c$, the project is producing. If, irrespective of changes in P , the firm had no choice but to continue producing throughout the future, the present value of the future flow of profits would be given by $P/\delta - c/r$. However, should P fall, the

¹⁸By substituting (17) for $V(P)$ into (15), the reader can check that β_1 and β_2 are the solutions to the following quadratic equation:

$$(1/2)\sigma^2\beta_1(\beta_1-1) + (r-\delta)\beta_1 - r = 0$$

Since $V(0) = 0$, the positive solution ($\beta_1 > 1$) must apply when $P < c$, and the negative solution ($\beta_2 < 0$) must apply when $P > c$. Note that β_1 is the same as β in eqn. (8).

firm can stop producing and avoid losses. The value of its option to stop producing is $A_2 P^{\beta_2}$.

A numerical example will help to illustrate this solution. Unless otherwise noted, we set $r = .04$, $\delta = .04$, and $c = 10$. Figure 6 shows $V(P)$ for $\sigma = 0$, $.2$, and $.4$. (Again, this is a conservative range of values for σ . For many commodities, the standard deviation of annual price changes is in excess of 40 percent.) Note that when $\sigma = 0$, there is no possibility that P will rise in the future, so the firm will never produce (and has no value) unless $P > 10$. If $P > 10$, $V(P) = (P - 10)/.04 = 25P - 250$. However, if $\sigma > 0$, the firm always has some value as long as $P > 0$; although the firm may not be producing today, it is likely to produce at some point in the future. Also, since the upside potential for future profit is unlimited while the downside is limited to zero, the greater is σ , the greater is the expected future flow of profit, and the higher is V .

Figure 7 shows $V(P)$ for $\sigma = .2$ and $\delta = .02$, $.04$, and $.08$. For any fixed risk-adjusted discount rate, a higher value of δ means a lower expected rate of price appreciation, and hence a lower value for the firm.

The Investment Decision.

Now that we have determined the value of this project, we must find the optimal investment rule. Specifically, what is the value of the firm's option to invest as a function of the price P , and at what critical price P^* should the firm exercise that option by spending an amount I to purchase the project?

By going through the same steps as above, the reader can check that the value of the firm's option to invest, $F(P)$, must satisfy the following differential equation:

$$(1/2)\sigma^2 P^2 F_{PP} + (r - \delta) P F_P - r F = 0 \quad (18)$$

$F(P)$ must also satisfy the following boundary conditions:

$$F(0) = 0 \quad (19a)$$

$$F(P^*) = V(P^*) - I \quad (19b)$$

$$F_P(P^*) = V_P(P^*) \quad (19c)$$

These conditions can be interpreted in the same way as conditions (6a)-(6c) for the model presented in Section 3. The difference is that the payoff from the investment, V , is now a function of the price P .

The solution to eqn. (18) and boundary condition (19a) is:

$$F(P) = \begin{cases} aP^{\beta_1} & , \quad P \leq P^* \\ V(P) - I & , \quad P > P^* \end{cases} \quad (20)$$

where β_1 is given above under eqn. (17). To find the constant a and the critical price P^* , we use boundary conditions (19b) and (19c). By substituting eqn. (20) for $F(P)$ and eqn. (17) for $V(P)$ (for $P \geq c$) into (19b) and (19c), the reader can check that the constant a is given by:

$$a = \frac{\beta_2 A_2}{\beta_1} (P^*)^{(\beta_2 - \beta_1)} + \frac{1}{\delta \beta_1} (P^*)^{(1 - \beta_1)} \quad (21)$$

and the critical price P^* is the solution to:

$$\frac{A_2(\beta_1 - \beta_2)}{\beta_1} (P^*)^{\beta_2} + \frac{(\beta_1 - 1)}{\delta \beta_1} P^* - \frac{c}{r} - I = 0 \quad (22)$$

Eqn. (22), which is easily solved numerically, gives the optimal investment rule. The reader can check that (22) has a unique positive solution for P^* that is larger than c . Hence the project must be more than profitable before it is optimal to invest in it. (The reader can also check that $V(P^*) > I$, so that the project must have an NPV that exceeds zero before it is optimal to invest.)

This solution is shown graphically in Figure 8, for $\sigma = .2$, $\delta = .04$, and $I = 100$. The figure plots $F(P)$ and $V(P) - I$. Note from boundary

condition (19b) that P^* satisfies $F(P^*) = V(P^*) - I$, and note from boundary condition (19c) that P^* is at a point of tangency of the two curves.

The comparative statics for changes in σ or δ are interesting here. As we say before, an increase in σ results in an increase in $V(P)$ for any P . (Remember that the project is essentially a set of call options on future production, and the greater the volatility of price, the greater the value of these options.) But although an increase in σ raises the value of the project, it also increases the critical price at which it is optimal to invest, i.e., $\partial P^*/\partial \sigma > 0$. The reason is that for any P , the opportunity cost of investing, $F(P)$, increases even more than $V(P)$. Hence as with the simpler model presented in the previous section, increased uncertainty reduces investment. This result is illustrated in Figure 9, which shows $F(P)$ and $V(P) - I$ for $\sigma = 0, .2$, and $.4$. When $\sigma = 0$, the critical price is 14, which just makes the value of the project equal to its cost of 100. As σ is increased, both $V(P)$ and $F(P)$ increase; P^* is 23.8 for $\sigma = .2$ and 34.9 for $\sigma = .4$.

An increase in δ also increases the critical price P^* at which the firm should invest. There are two opposing effects here. If δ is larger, so that the expected rate of increase of P is smaller, options on future production are worth less, so $V(P)$ is smaller. At the same time, the expected rate of growth of $F(P)$ is smaller, so there is more incentive to exercise the investment option, rather than keep it alive. The first effect outweighs the second, so that a higher δ implies a higher P^* . This is illustrated in Figure 10, which shows $F(P)$ and $V(P) - I$ for $\delta = .04$ and $.08$. Note that when δ is increased, $V(P)$ and hence $F(P)$ fall sharply, and the tangency at P^* moves to the right.

This result might at first seem to contradict what the simpler model of Section 3 tells us. Recall that in that model, an increase in δ reduces the critical value of the project, V^* , at which the firm should invest. But although in this model P^* is higher when δ is larger, the corresponding value of the project, $V(P^*)$, is lower. This can be seen from Figure 11, which shows P^* as a function of σ for $\delta = .04$ and $.08$, and Figure 12, which shows $V(P^*)$. If, say, σ is $.2$ and δ is increased from $.04$ to $.08$, P^* will rise from 23.8 to 29.2 , but even at the higher P^* , V is lower. Thus $V^* - V(P^*)$ is declining with δ , just as in the simpler model.

This model, although still unrealistic in some respects, gets at the essence of how uncertainty over future prices affects the value of a project and the decision to invest in the project. Furthermore, the model has practical application, especially if the project is one that produces a traded commodity, like copper or oil. In that case, σ and δ can be determined directly from futures and spot market data. Also, the model can easily be expanded to allow for fixed costs of temporarily stopping and restarting production, if such costs are important.¹⁹

Alternative Stochastic Processes.

The model can also be adapted to allow for alternative stochastic processes for the price, P , although in most cases numerical methods will then be necessary to obtain a solution. For example, for some commodities there may be reason to believe that P follows a mean-reverting process, such as:

$$dP/P = \lambda(\bar{P} - P)dt + \sigma dz \quad (23)$$

¹⁹See Brennan and Schwartz (1985) for a model of a mining project that does just this.

Here, P tends to revert back to a "normal" level \bar{P} . (For example, \bar{P} might be long-run marginal cost in the case of commodity like copper or coffee.) By going through the same arguments as we did before, it is easy to show that in this case $V(P)$ must satisfy the following differential equation:

$$(1/2)\sigma^2 P^2 V_{PP} + [(r-\mu-\lambda)P + \lambda\bar{P}]PV_P - rV + j(P-c) = 0 \quad (24)$$

with boundary conditions (16a) - (16d). The value of the investment option, $F(P)$, must satisfy:

$$(1/2)\sigma^2 P^2 F_{PP} + [(r-\mu-\lambda)P + \lambda\bar{P}]PF_P - rF = 0 \quad (25)$$

and boundary conditions (19a) - (19c). Equations (24) and (25) are ordinary differential equations, and solution by numerical methods is reasonably straightforward.

5. Empirical Evidence.

The models discussed above are theoretical, but have implications for investment behavior. Most important, they suggest that investment should be particularly sensitive to uncertainty of various forms, more sensitive than standard models would imply. It is interesting, then, that most econometric models of aggregate economic activity ignore the role of risk, or deal with it only implicitly. A more explicit treatment of risk may help to better explain economic fluctuations, and especially investment spending.

Consider the recessions of 1975 and 1980. The sharp jumps in energy prices that occurred in 1974 and 1979-80 clearly contributed to those recessions. They caused a reduction in the real national incomes of oil importing countries, and they led to "adjustment effects" -- inflation and a further drop in real income and output resulting from the rigidities that prevented wages and non-energy prices from coming into equilibrium quickly. But those energy shocks also caused greater uncertainty over future economic

conditions. For example, it was unclear whether energy prices would fall or continue to rise, what the impact of higher energy prices would be on the marginal products of various types of capital, how long-lived the inflationary impact of the shocks would be, etc. Other events also made the economic environment more uncertain, especially in 1979-82: much more volatile exchange rates and interest rates. This may have contributed to the decline in investment spending that occurred during these periods.²⁰

Explaining Investment Behavior.

The explanation of aggregate and sectoral investment behavior remains problematic in the development of macro-econometric models. Existing models have had, at best, limited success in explaining or predicting investment. The problem is not simply that these models have been able to explain and predict only a small portion of the movements in investment. In addition, constructed quantities that in theory should have strong explanatory power -- e.g. Tobin's q , or various measures of the cost of capital -- in practice do not, and leave much of investment spending unexplained.²¹

It is easy to think of reasons for the failings of these models. If nothing else, there are formidable estimation problems arising from aggregation (across firms, and across investment projects of different gestations). At issue is the importance of their failure to treat risk properly. Effects of risk are typically handled by assuming that a risk premium (obtained, say, from the CAPM) can be added to the discount rate used to calculate the present value of a project. But as we have learned

²⁰This point was made by Bernanke (1983), particularly with respect to changes in oil prices. Also, see Evans (1984) and Tatom (1984) for a discussion of the depressive effects of increased interest rate volatility.

²¹See Kopcke (1985) for an overview, as well as examples and comparisons of traditional approaches to modelling investment spending.

from financial option pricing and its application to real investment, the correct discount rate cannot be obtained without actually solving the option valuation problem, that discount rate need not be constant over time, and it will not equal the firm's average cost of capital. As a result, simple cost of capital measures, based on rates of return (simple or adjusted) to equity and debt, may be poor explanators of investment spending.

This can be seen in the context of models based on Tobin's q . A good example is the model of Abel and Blanchard (1986), which is one of the most sophisticated attempts to explain investment in a q theory framework; it uses a carefully constructed measure for marginal rather than average q , incorporates delivery lags and costs of adjustment, and explicitly models expectations of future values of explanatory variables.

The model is based on the standard discounted cash flow rule, "invest in the marginal unit of capital if the present discounted value of the expected flow of profits resulting from the unit is at least equal to the cost of the unit." Let $\pi_t(K_t, I_t)$ be the maximum value of profits at time t , given the capital stock K_t and investment level I_t , i.e. it is the value of profits assuming that variable factors are used optimally. It depends on I_t because of costs of adjustment; $\partial\pi/\partial I < 0$, and $\partial^2\pi/\partial I^2 < 0$, i.e. the more rapidly new capital is purchased and installed, the more costly it is. Then the present value of current and future profits is given by:

$$V_t = E_t \left[\sum_{j=0}^{\infty} \left[\prod_{i=0}^j (1+R_{t+i})^{-1} \right] \pi_{t+j}(K_{t+j}, I_{t+j}) \right]$$

where E_t denotes an expectation, and R is the discount rate. Maximizing this with respect to I_t , subject to the condition $K_t = (1-\delta)K_{t-1} + I_t$ (where δ is the rate of depreciation), gives the following marginal condition:

$$-E_t(\partial\pi_t/\partial I_t) = q_t, \quad (26)$$

$$\text{where } q_t = E_t \left[\sum_{j=0}^{\infty} \left[\prod_{i=0}^j (1+R_{t+i})^{-1} \right] (\partial\pi_{t+j}/\partial K_{t+j})(1-\delta)^j \right] \quad (27)$$

In other words investment occurs up to the point where the cost of an additional unit of capital equals the present value of the expected flow of incremental profits resulting from the unit. Abel and Blanchard estimate both linear and quadratic approximations to q_t , and use vector autoregressive representations of R_t and $\partial\pi_t/\partial K_t$ to model expectations of future values. Their representation of R_t is based on a weighted average of the rates of return on equity and debt.

If the correct discount rates R_{t+i} were known, eqns. (26) and (27) would indeed accurately represent the optimal investment decision of the firm. The problem is that these discount rates are usually not known, and generally will not equal the average cost of capital of the firm, or some related variable. Instead, these discount rates can only be determined as part of the solution to the firm's optimal investment problem. This involves valuing the firm's options to make (irreversible) marginal investments (now or in the future), and determining the conditions for the optimal exercise of those options. Thus the solution to the investment problem is more complicated than the first-order condition given by (26) and (27) would suggest.

As an example, consider a project that has zero systematic (non-diversifiable) risk. The use of a risk-free interest rate for R would lead to much too large a value for q_t , and might suggest that an investment expenditure should be made, whereas in fact it should be delayed. Furthermore, there is no simple way to adjust R properly. The problem is that the calculation ignores the opportunity cost of exercising the option to

invest. This may be why Abel and Blanchard conclude that "our data are not sympathetic to the basic restrictions imposed by the q theory, even extended to allow for simple delivery lags."

Does Irreversibility Matter?

Unfortunately, incorporating irreversibility into models of aggregate investment spending is not a simple matter. As we have seen, the equations describing optimal investment decisions are extremely nonlinear, even for very simple models. Partly as a result of this, there has been very limited empirical work to date that tests the importance of irreversibility for the modelling of investment spending.

One paper that does provide such a test is by Bizer and Sichel (1988). They develop a model of capital accumulation and utilization with asymmetric costs of adjustment, i.e., the costs of adjusting the capital stock up or down can differ. If irreversibility is important, one would expect to find that downward adjustment costs exceed upward ones. Bizer and Sichel derive an Euler equation, which they estimate using Hansen and Singleton's generalized instrumental variable procedure. They do not have firm data, so instead they use 2-digit SIC industry data. They measure asymmetry of adjustment costs with respect to two reference points: a zero level of investment, and a "normal" (average) level of investment.

Their preliminary results indicate some evidence of irreversibility, in particular in primary metals, fabricated metal products, and possibly the paper industry. But they also find that upward adjustment costs exceed downward ones in the food and petroleum industries. This may simply mean that aggregation is masking irreversibility. Other problems include the use of a single discount factor (the S&P dividend/price ratio, which is very volatile) and cost of capital for all industries. Nonetheless, their

approach seems like a promising way to test for effects of irreversibility, particularly if used with disaggregated data. It does not, however, explicitly deal with effects of changes in risk.

In a recent working paper (1986), I performed some very simple non-structural tests for the importance of risk. I used data on the stock market, on the grounds that when product markets become more volatile, we would expect stock prices to also become more volatile, so that the variance of stock returns will be larger. This was indeed the case, for example, during the recessions of 1975 and 1980, and most dramatically during the Great Depression. Thus the variance of aggregate stock returns should be correlated with aggregate product market uncertainty.

Stock returns themselves are also a predictor of aggregate investment spending. My concern was whether the variance of stock returns also has predictive power with respect to investment, and whether that predictive power goes beyond that of stock returns themselves, as well as other variables that would usually appear in an empirical investment equation. I conducted two related exploratory tests. First, I tested and was able to accept the hypothesis that the variance of stock returns Granger-cause the real growth rate of investment. Specifically, the variance of returns is a strong predictor of investment growth, but investment growth does not predict the variance of returns.²²

²²To say that "X causes Y," two conditions should be met. First, X should help to predict Y, i.e. in a regression of Y against past values of Y, the addition of past values of X as independent variables should contribute significantly to the explanatory power of the regression. Second, Y should not help to predict X. (If X helps to predict Y and Y helps to predict X, it is likely that one or more other variables are in fact "causing" both X and Y.)

Second, I ran a set of simple regressions of the growth rate of investment against stock returns, the variance of stock returns, and a set of additional explanatory variables that usually appear in empirical investment equations. In these regressions, I found the variance of stock returns to be an important explainer of the growth rate of investment spending, in most cases the most significant single explainer.

Of course these tests are extremely crude, and based on aggregate data and what is probably a very imperfect measure of risk. (Even if the variance of stock returns is a good proxy for the volatility of cash flows, it does not capture the "peso problem," i.e., the risk associated with some possible future catastrophic event.) Nonetheless, the results suggest that the explicit inclusion of market risk measures may help to improve our ability to explain and predict investment spending, and that the development of structural models that include such measures should be an important research priority.

6. Policy Implications and Future Research.

Much of our discussion has dealt with decision rules for individual firms. Now we will turn our attention to what these rules might imply about policy at the industry-wide or economy-wide level. Most important is the fact that we would expect investment spending on an aggregate level to be especially sensitive to risk in various forms: uncertainties over the future product prices and operating costs that determine cash flows, uncertainty over future interest rates, and uncertainty over the cost and timing of the investment itself. This means that if a goal of macroeconomic policy is to stimulate investment, stability and credibility may be much more important than tax incentives or interest rates. Put another way, if uncertainty over

the economic environment is high, tax and related incentives may have to be very large to have any significant impact on investment.

The role of interest rates and interest rate stability are especially interesting here. In a recent paper, Ingersoll and Ross (1988) examined irreversible investment decisions when the interest rate evolves stochastically, but future cash flows are known with certainty. As with uncertainty over future cash flows, this creates an opportunity cost of investing, so that the traditional NPV rule will accept too many projects. Instead, an investment should be made only when the interest rate is below a critical rate, r^* , which is lower than the internal rate of return, r^0 , which makes the NPV zero. Furthermore, the difference between r^* and r^0 grows as the volatility of interest rates grows.

Ingersoll and Ross also show that for long-lived projects, a decrease in expected interest rates for all future periods need not accelerate investment. The reason is that such a change also lowers the cost of waiting, and thus can have an ambiguous effect on investment. This is another example of how the level of interest rates may be of only secondary importance as a determinant of aggregate investment spending. Interest rate volatility (as well as the volatility of other variables) may be more important. Rough empirical tests (such as those in my 1986 paper) might provide some evidence on whether this is indeed the case.

The irreversibility of investment also helps to explain why trade reforms can turn out to be counterproductive, with a liberalization leading to a decrease in aggregate investment. As Dornbusch (1987) and Van Wijnbergen (1985) have noted, uncertainty over future tariff structures, and hence over future factor returns, creates an opportunity cost to committing capital to new physical plant. Foreign exchange and liquid assets held

abroad involve no such commitment, and so may be preferable even though the expected rate of return is lower.²³ Likewise, it may be difficult to stem or reverse capital flight if there is a perception that it may become more difficult to take capital out of the country than to bring it in.

The irreversibility of investment is also likely to have policy implications for specific industries. The energy industry is an example. In that industry, an aspect of stability and credibility has to do with the threat of price controls, "windfall" profit taxes, or related policies that might be imposed should prices rise substantially. Given that price is evolving stochastically, how should investment decisions take into account a probability that price may be capped at some level?²⁴ How does the threat of a price cap, or the threat of outright expropriation, affect investment behavior, and industry capacity and production? This is a problem that is amenable to the methodology described in this paper, and might be addressed as a topic of future research.

A more fundamental problem of stability is the volatility of market prices themselves. For many raw commodities (oil is an obvious example), price volatility rose substantially in the early 1970's, and has been high

²³Van Wijnbergen is incorrect, however, when he claims (p. 369) that "there is only a gain to be obtained by deferring commitment if uncertainty decreases over time so that information can be acquired about future factor returns as time goes by." Van Wijnbergen bases his analysis on the models of Bernanke (1983) and Cukierman (1980), in which there is indeed a reduction in uncertainty over time. But as we have seen from the models discussed in Sections 3 and 4 of this paper, there is no need for uncertainty over future conditions to fall over time. In those models, the future value of the project or price of output is always uncertain, but there is nonetheless an opportunity cost to committing resources.

²⁴Dixit (1988) has addressed this problem, but only in a very preliminary way. He develops a model of investment in which the free market price is stochastic, but there is a permanent ceiling on the price that the firm can charge.

since. Other things equal, we would expect this to increase the value of land and other resources needed to produce the commodity, but have a depressing effect on construction expenditures and production capacity. This means that there may be added benefits from policies that stabilize prices.

Most studies of the gains from price stabilization focus on adjustment costs and the curvature of demand and (static) supply curves. (See Newbery and Stiglitz (1981) for an overview.) The irreversibility of investment creates an additional gain which may be substantial. Krugman (1988) has examined this in a limited way in the context of exchange rate stabilization. He showed how irreversibility (i.e., sunk costs) will cause firms to remain in markets even when they are losing money because of adverse exchange rate movements, and fail to enter markets even though favorable exchange rate movements would seem to make entry profitable. Furthermore, this feeds back to exchange rates themselves, and adds to their volatility. In a similar vein, Caballero and Corbo (1988) have shown how uncertainty over future real exchange rates can depress exports.

The emerging literature on these effects of uncertainty and instability has produced models that are fairly simple and almost purely theoretical. While it is clear that increase in the volatility of, say, interest rates or exchange rates should depress investment, it is not at all clear how large the effect is likely to be. Nor is it clear how important these factors have been as explanators of investment across countries and over time.

Determining the importance of these factors should be a research priority. One approach is to do empirical testing of the sort discussed in the preceding section, perhaps using cross section data for a number of countries. Another approach is to construct a simulation model, the

structure of which might be similar to the model presented in Section 4, and parameterize it so that it "fits" a particular industry. One could then calculate predicted effects of observed changes in, say, price volatility, and compare them to the predicted effects of changes in interest rates or tax rate. Simulation models of this sort could likewise be constructed to predict the effect of a perceived possible shift in the tax regime, the imposition of price controls, etc. Such models may also be a good way to study uncertainty of the "peso problem" sort.

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FIGURE 1 - PRICE OF WIDGETS

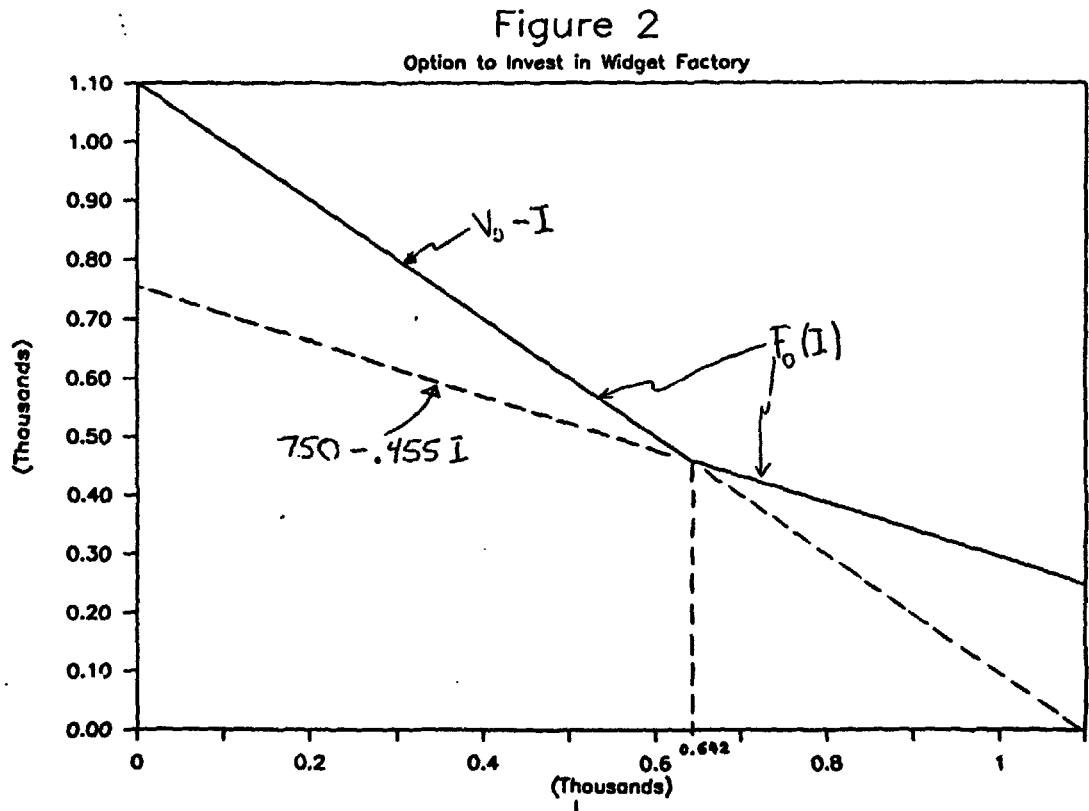
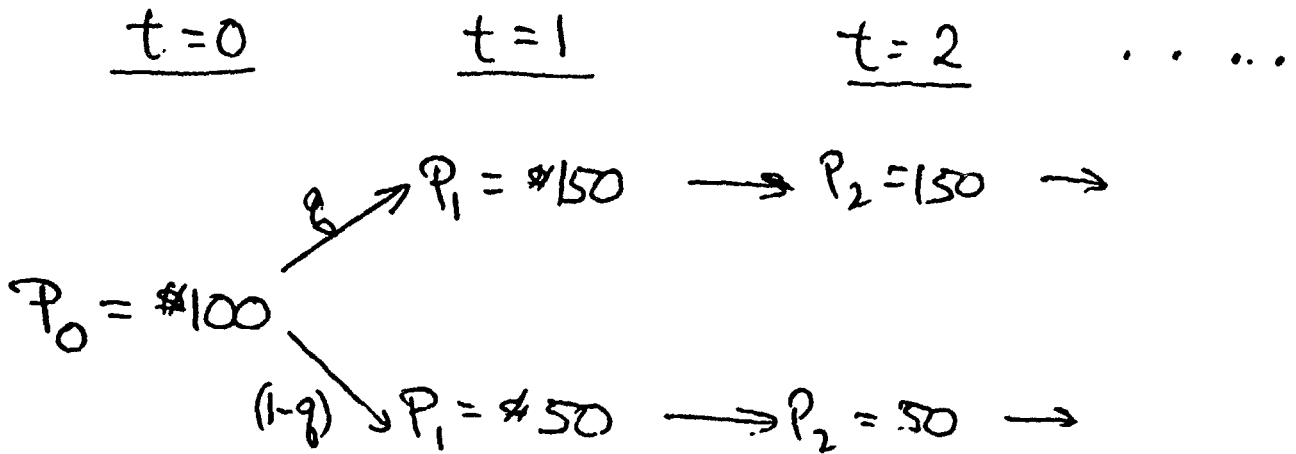
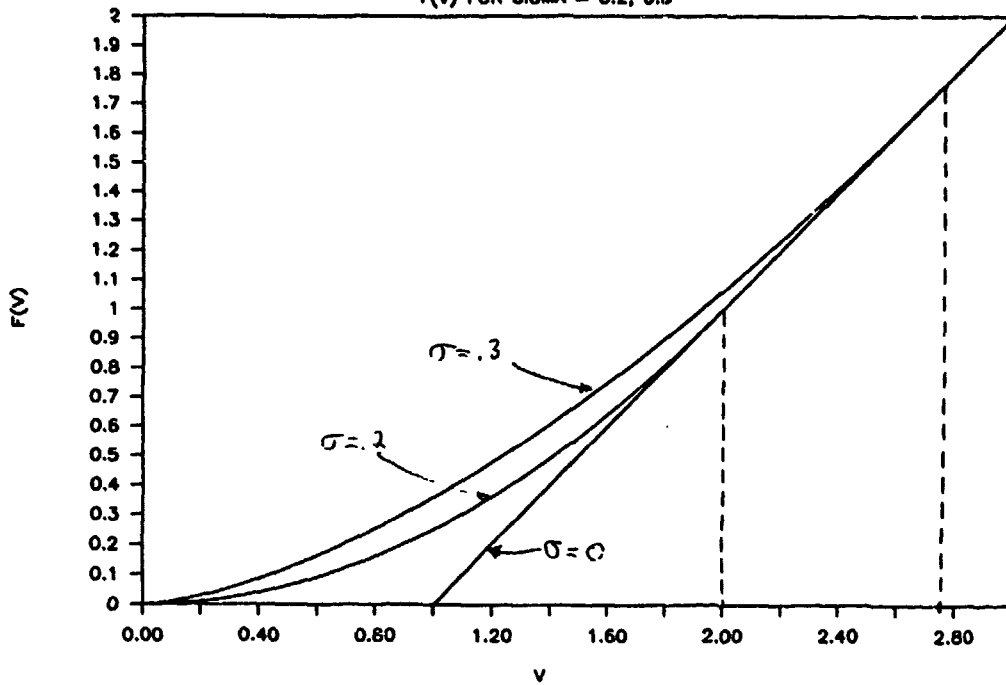


Figure 3

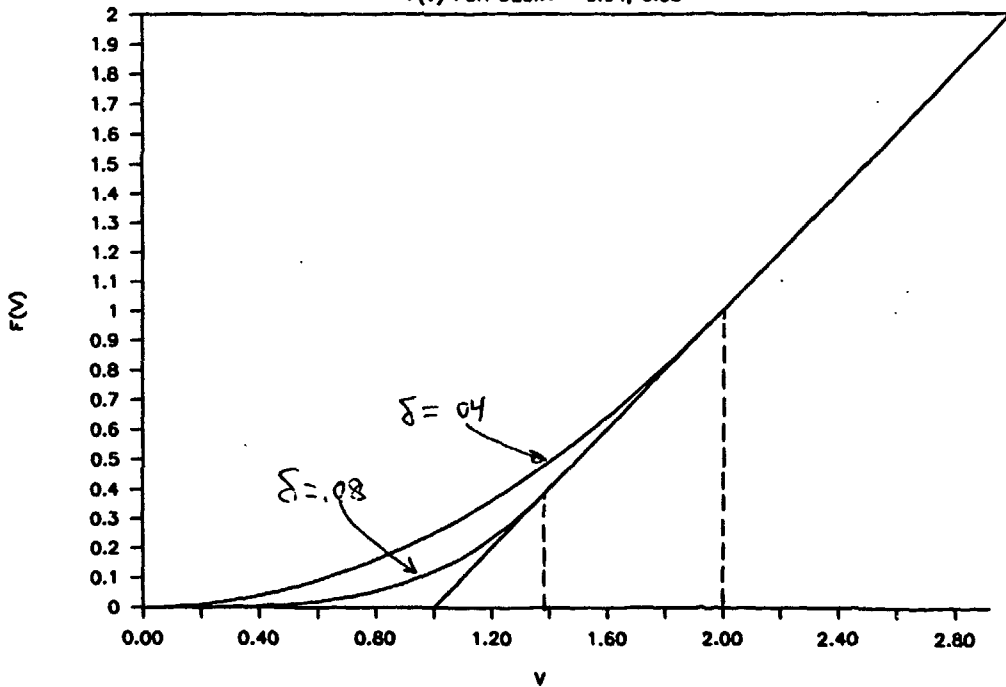
$F(V)$ FOR $\sigma = 0.2, 0.3$



NOTE: $\Delta = 0.04$, $r = 0.04$, $I = 1$

Figure 4

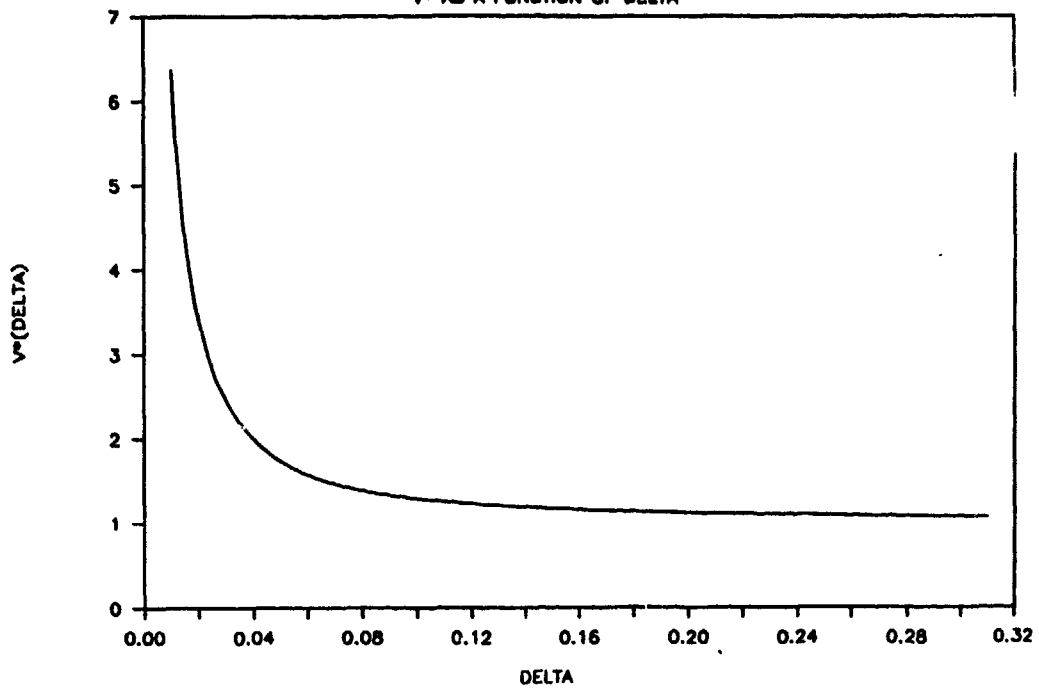
$F(V)$ FOR $\Delta = 0.04, 0.08$



NOTE: $\sigma = 0.2$, $r = 0.04$, $I = 1$

Figure 5

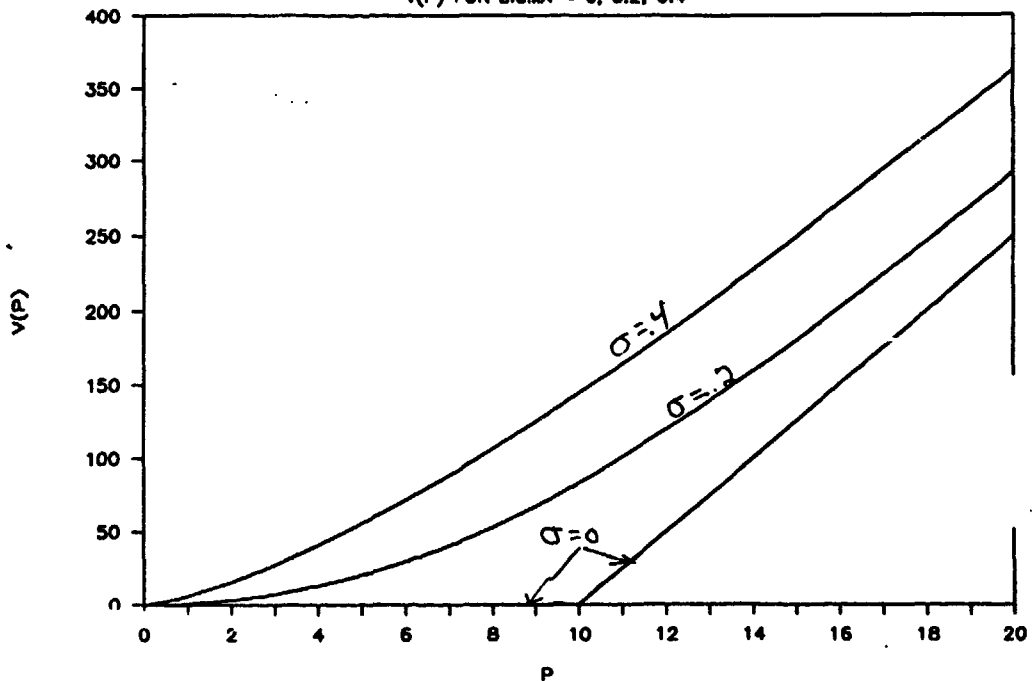
V^* AS A FUNCTION OF DELTA



NOTE: $\Sigma = 0.2$, $r = 0.04$, $I = 1$

Figure 6

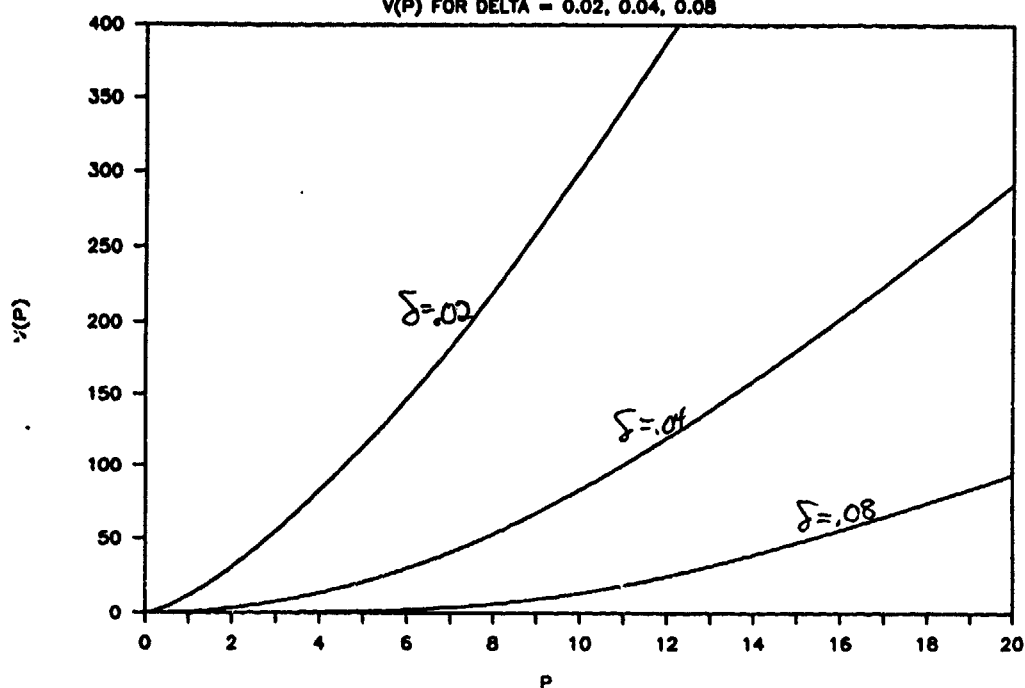
$V(P)$ FOR $\Sigma = 0, 0.2, 0.4$



Note: $\delta = .04$, $r = .04$, $C = 10$

Figure 7

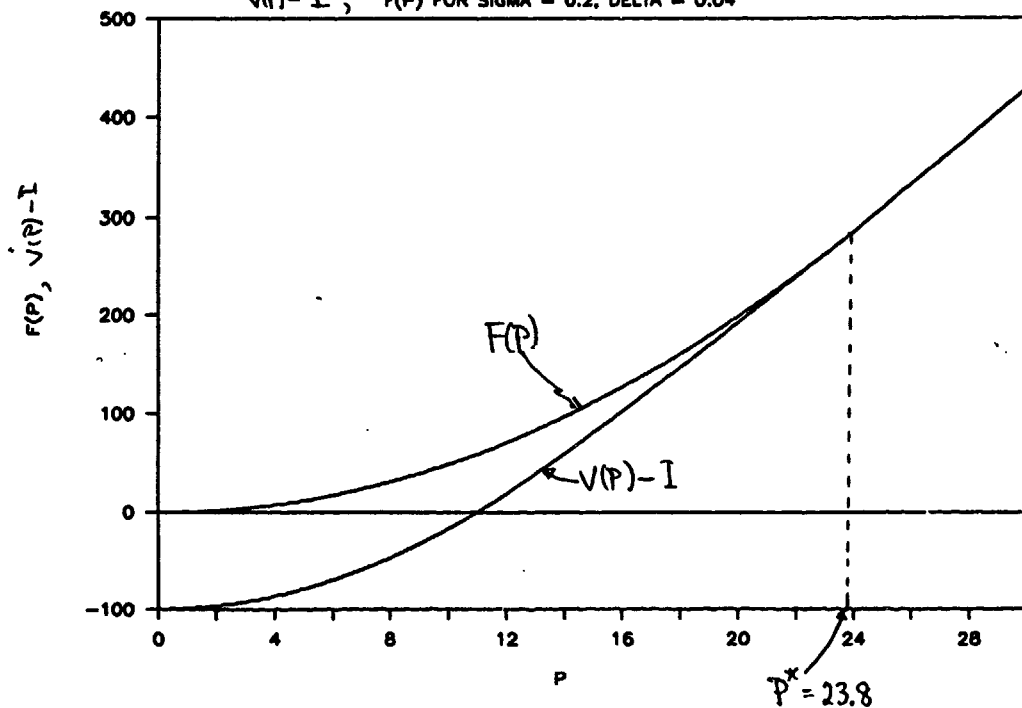
$V(P)$ FOR $\Delta = 0.02, 0.04, 0.08$



Note: $r = .04$, $\sigma = .2$, $c = 10$

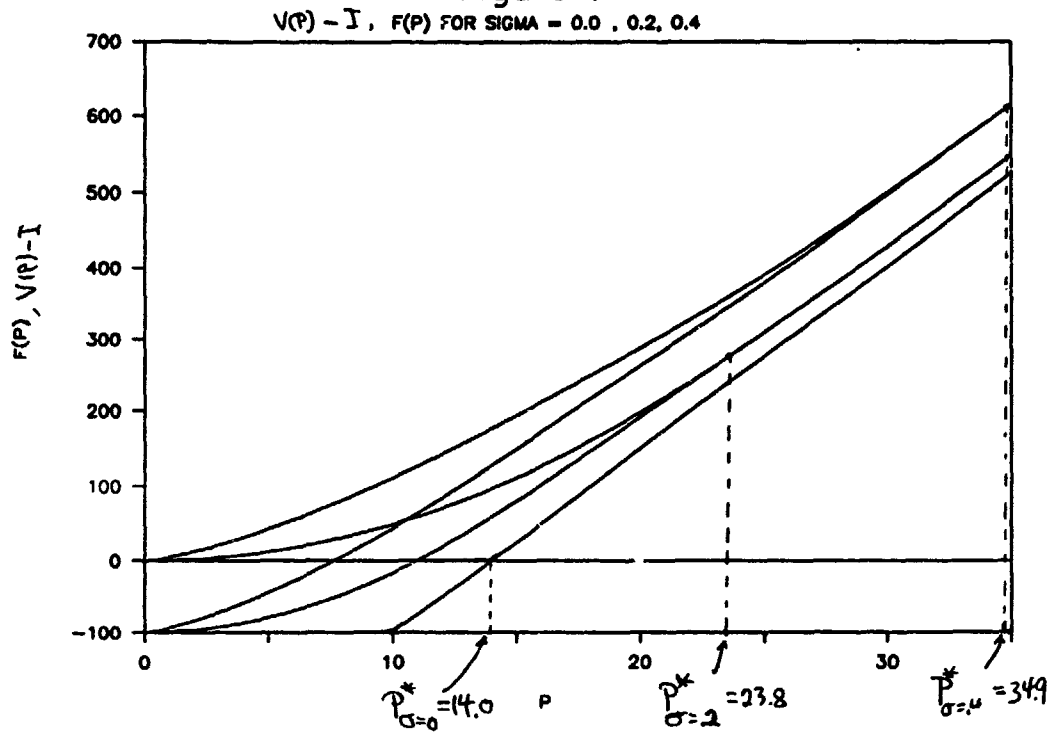
Figure 8

$V(P) - I$, $F(P)$ FOR $\sigma = 0.2$, $\Delta = 0.04$



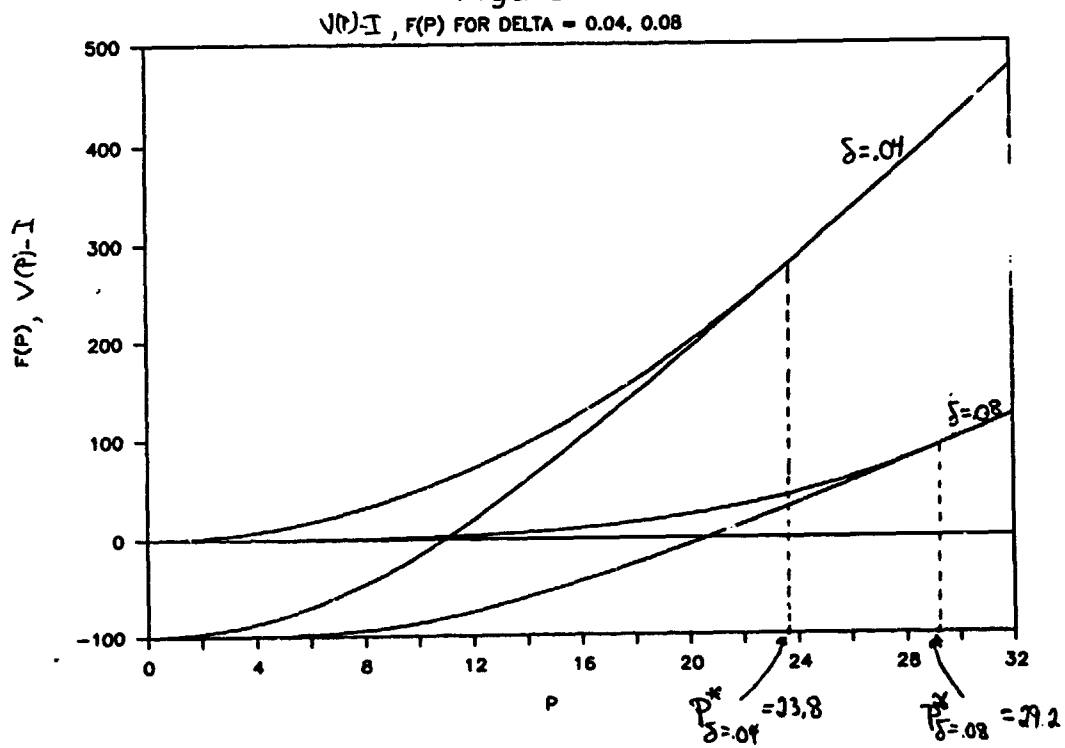
Note: $r = .04$, $c = 10$, $I = 100$

Figure 9



Note: $r = .04, \delta = .04, c = 10, I = 100$

Figure 10



Note: $r = .04, \sigma = .2, c = 10, I = 100$

Figure 11

P^* VS. SIGMA FOR DELTA = 0.04, 0.08

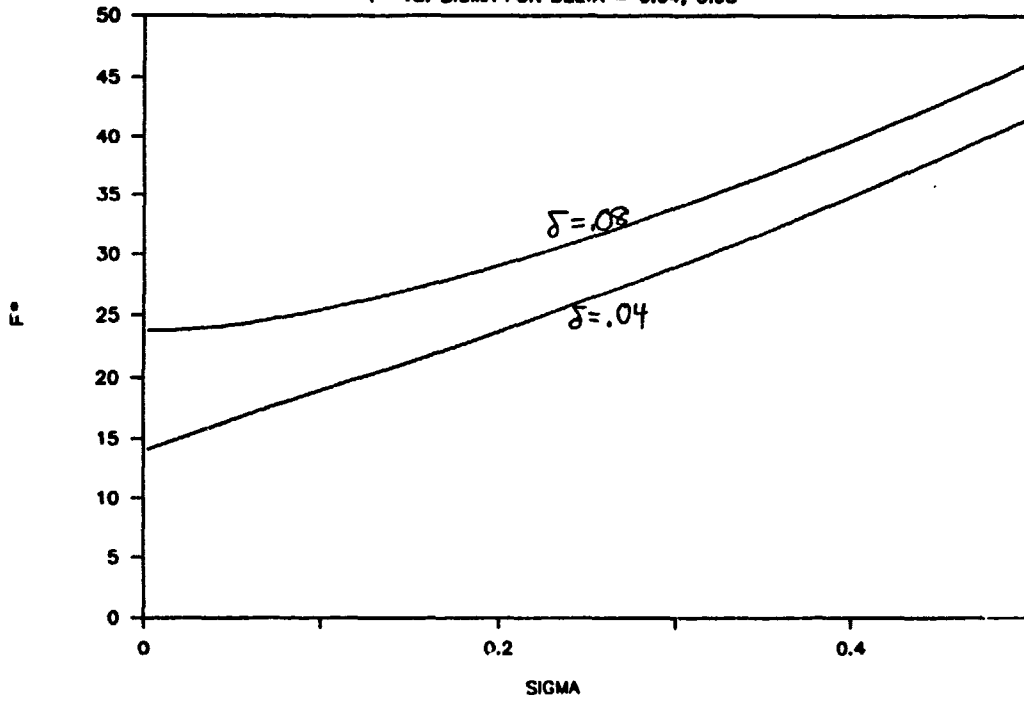
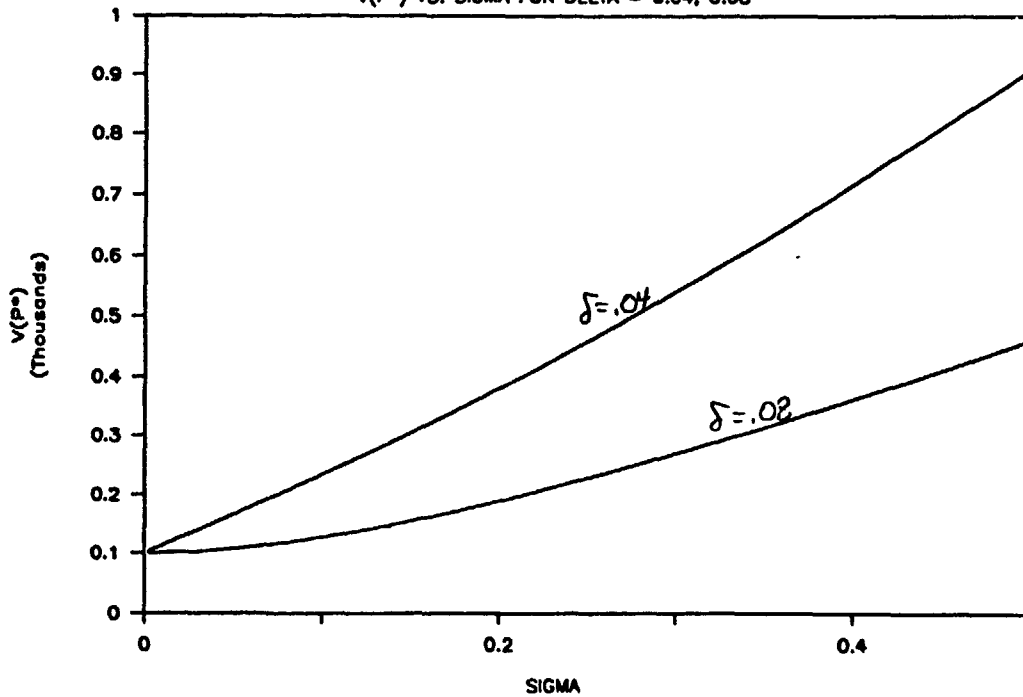


Figure 12

$V(P^*)$ VS. SIGMA FOR DELTA = 0.04, 0.08



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